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BOUNDEDNESS OF THE SPECTRA OF BASE FUNCTIONS FOR NEW SPACES

Hee Chul Pak

Abstract. We investigate the boundedness of the spectrum of a convex base function for a new function space. The result guarantees the continuity of the Calderón-Zygmund operators on the new space.

1. Introduction

We have built up a new function space in order to generalize the classical Lebesgue spaces [2, 3, 4, 5, 6]. The motivation of this research stems from taking a close look at the L^p-norm: $\|f\|_{L^p} = \left(\int_X |f(x)|^p \, d\mu\right)^{1/p}$ of the Lebesgue spaces $L^p(X)$, $1 \le p < \infty$. It can be rewritten as

$$\|f\|_{L^p} := \alpha^{-1}\left(\int_X \alpha(|f(x)|)\, d\mu\right) \tag{1.1}$$

with the base function α as

$$\alpha(x) := x^p.$$

The main point of this research is to replace base functions α with various base functions which do not hurt the beauty of L^p-norm (1.1) too much.

We investigate the continuity and the discontinuity of the Calderón-Zygmund operators on the new function spaces equipped with various base functions. For this, in [6], we introduced the concept of the *spectrum* (or *exponent function*) p_α of a base function α defined as

$$p_\alpha(x) := x\frac{\alpha'(x)}{\alpha(x)}. \tag{1.2}$$

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For example, the Lebesgue base function $\alpha(x) = x^p$ has a point spectrum: $p_\alpha(x) = p$ with $1 \le p < \infty$. The spectrum has been a touchstone of identifying the continuity of the singular integrals on those spaces.

In [6], we figure out that the Calderón-Zygmund operators pertain their continuity on this function space if it has a bounded spectrum which is far off from the value 1, that is to say, there exist some constants c_1, c_2 satisfying

$$1 < c_1 \le p_\alpha(x) \le c_2 < \infty \tag{1.3}$$

for almost every $x > 0$. On the contrary, it is noticed that some singular integrals of the Calderón-Zygmund type operators may fail to be continuous on the space if we suppose either 1 or ∞ is accumulated by the spectrum p_α.

In this paper, we present the boundedness (1.3) of the spectrum for the convex Hölder base functions. Our result together with the Marcinkiewicz type interpolation theorem in [6] guarantees the continuity of the Calderón-Zygmund operators on those spaces equipped with the conjugate pair of the convex Hölder base functions.

In the following, $(X, \mathfrak{M}, \mu)$ represents an abstract measure space and $\bar{\mathbb{R}}_+ = \{x \in \mathbb{R} : x \ge 0\}$.

2. Hölder base functions

Notions of Hölder functions have been developed to find appropriate base functions that permit the Hölder's inequality. In this section, we briefly introduce the fundamentals of the Hölder functions - the details can be found in [2, 5, 6].

A *pre-Hölder function* $\alpha : \bar{\mathbb{R}}_+ \to \bar{\mathbb{R}}_+$ is an *absolutely continuous bijective* function satisfying $\alpha(0) = 0$. If there exists a pre-Hölder function β satisfying

$$\alpha^{-1}(x)\beta^{-1}(x) = x \tag{2.1}$$

for all $x \in \bar{\mathbb{R}}_+$, then β is called the *conjugate* (*pre-Hölder*) *function* of α. In the relation (2.1), the notations α^{-1}, β^{-1} are the inverse functions of α, β, respectively. Some examples of pre-Hölder pairs are: $(\alpha(x), \beta(x)) = (x^p, x^q)$ for $p > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, and

$$(\alpha, \beta) := (\lambda \circ A, \lambda \circ \tilde{A}) \tag{2.2}$$

where we set $\lambda(x) = A^{-1}(x)\tilde{A}^{-1}(x)$ for any Orlicz N-function A together with its complementary N-function $\tilde{A}$.

The *spectrum* (or *exponent function*) p_α of a pre-Hölder function α is defined as

$$p_\alpha(x) := x\frac{\alpha'(x)}{\alpha(x)}. \tag{2.3}$$

A pre-Hölder function α permits a conjugate function if and only if α satisfies the limit conditions:

$$\lim_{x\to 0+}\frac{\alpha(x)}{x} = 0, \quad \lim_{x\to\infty}\frac{\alpha(x)}{x} = \infty \tag{2.4}$$

together with the spectrum condition:

$$p_\alpha(x) > 1 \tag{2.5}$$

for almost every $x > 0$. Also, for (α, β) a pre-Hölder pair, we have

$$\frac{1}{p_\alpha}(s) + \frac{1}{p_\beta}(t) = 1, \qquad \alpha(s) = \beta(t).$$

For details, we refer [6].

Let Φ be a two-variable function on $\bar{\mathbb{R}}_+ \times \bar{\mathbb{R}}_+$ defined by:

$$\Phi(x,y) := \alpha^{-1}(x)\beta^{-1}(y).$$

Then we observe that the equation of the tangent plane T of Φ at a point $(\alpha(a), \beta(b))$ is represented by

$$T(x,y) = \frac{1}{p_\alpha}\frac{ab}{\alpha(a)}\,x + \frac{1}{p_\beta}\frac{ab}{\beta(b)}\,y + ab\,\theta_f \tag{2.6}$$

with $\theta_f = 1 - \frac{1}{p_\alpha} - \frac{1}{p_\beta}$. This motivation leads to define the Hölder functions as follows.

Definition 2.1. *Let $\hbar > 0$ be given. A pre-Hölder function α with the conjugate function β is said to be a Hölder function if for any positive constants $a, b > 0$, there exist constants θ_1, θ_2 and θ_f (depending on a and b) such that*

$$\theta_1 + \theta_2 + \theta_f \le \hbar$$

and that a dominating condition

$$\Phi(x,y) \le \theta_1\frac{ab}{\alpha(a)}\,x + \theta_2\frac{ab}{\beta(b)}\,y + ab\,\theta_f, \tag{2.7}$$

holds for all $(x,y) \in \bar{\mathbb{R}}_+ \times \bar{\mathbb{R}}_+$. A Hölder function α is defined as an s-Hölder function if we can choose $\theta_f = 0$ in (2.7).

For example, any (convex) function satisfying

$$\alpha(x) := \begin{cases} x^p & \text{for } 0 \leq x \leq 1 \\ x^q & \text{for sufficiently large } x \end{cases} \tag{2.8}$$

($1 < p, q < \infty$) is a Hölder function, and so are many variants of (2.8).

For a Hölder function α and a given integrable positive function w on X, we define

$$P_\alpha(X, w) := \{ f \mid f \text{ is a measurable function on } X \text{ satisfying } \|f\|_{P_\alpha} < \infty \},$$

where

$$\|f\|_{P_\alpha} := \alpha^{-1} \left(\int_X \alpha(|f(x)|) w(x) \, d\mu \right). \tag{2.9}$$

Without loss of generality, we may assume that the weighted function w can be chosen to be $\int_X w(x)dx = 1$.

Let α be a Hölder function and β be the corresponding Hölder conjugate function. Then for any $f \in P_\alpha(X, w)$ and $g \in P_\beta(X, w)$, we have

$$\left| \int_X f(x) g(x) w(x) \, d\mu \right| \leq \hbar \|f\|_{P_\alpha} \|g\|_{P_\beta}. \tag{2.10}$$

Hölder's inequality always incubates the Minkowski type inequality: for $f_1, f_2 \in P_\alpha(X, w)$, we obtain

$$\|f_1 + f_2\|_{P_\alpha} \leq \hbar \{ \|f_1\|_{P_\alpha} + \|f_2\|_{P_\alpha} \}. \tag{2.11}$$

Also, for any constant $k \geq 0$ and for $f \in P_\alpha(X, w)$, we have

$$\frac{k}{\hbar} \|f\|_{P_\alpha} \leq \|kf\|_{P_\alpha} \leq k\hbar \|f\|_{P_\alpha}.$$

In particular, when $\hbar = 1$, we have the homogeneity:

$$\|kf\|_{P_\alpha} = k \|f\|_{P_\alpha}.$$

Quasi-homogeneity of the norm is good enough to exploit estimates for the existence theory of nonlinear partial differential equations[2, 5] and to study singular integrals on these genealogical function spaces.

3. Boundedness of the spectrum

In this section we introduce the boundedness of the spectrum for a pair of convex base functions. The result implies the continuity of the Calderón-Zygmund operators on those function spaces.

THEOREM 3.1 (Boundedness of the spectrum). *Let (α, β) be a convex Hölder pair and let p_α be the spectrum of α. Then there exist some constants $1 < c_1$, $c_2 < \infty$ satisfying*

$$c_1 \leq p_\alpha(t) \leq c_2 \tag{3.1}$$

for almost every $t > 0$.

Proof. We first show that $\{p_\alpha(t)\}_{t>0}$ is bounded above. Since α is convex, we have

$$t\alpha'(t) \leq \int_t^{2t} \alpha'(s)ds \leq \int_0^{2t} \alpha'(s)ds = \alpha(2t) \qquad \text{for } t > 0.$$

Hence in order to find a constant $c_2 > 1$ satisfying $p_\alpha(t) \leq c_2$, it is enough to show that there is a constant $c_2 > 1$ such that $\alpha(2t) \leq c_2\alpha(t)$. We demonstrate this by a contradiction.

Suppose there exists a sequence $\{t_j\}_{j=1}^\infty$ of positive numbers such that

$$\alpha(2t_j) \geq 2^j \alpha(t_j) \tag{3.2}$$

for all $j = 1, 2, \cdots$. We drive a contradiction by violating the Minkowski type inequality (2.11) for the measurable functions defined, for example, on the Lebesgue measure space $(\mathbb{R}^n, \mathfrak{M}, \mu)$.

Let K be a compact subset of $\mathbb{R}^n$. We set

$$\|f\|_{P_\alpha(K)} := \alpha^{-1}\left(\int_K \alpha(|f(x)|)w(x)\,d\mu\right)$$

with a constant weighted function $w(x) := \frac{1}{\mu(K)}$, and choose a sequence of mutually disjoint, measurable subsets $\{K_j\}_{j=1}^\infty$ of K such that

$$\mu(K_j) = \frac{\alpha(1)\mu(K)}{2^j\alpha(t_j)}.$$

We define

$$f(x) := \sum_{j=1}^{\infty} t_j\,\chi_{K_j}(x), \quad x \in K$$

to obtain

$$\begin{aligned}\|f\|_{P_\alpha} &= \alpha^{-1}\left(\int_K \alpha(|f(x)|)\frac{d\mu}{\mu(K)}\right)\\ &= \alpha^{-1}\left(\frac{1}{\mu(K)}\sum_{j=1}^{\infty}\alpha(t_j)\mu(K_j)\right) = 1.\end{aligned}$$

On the other hand, from (3.2), one has

$$\begin{aligned}\|2f\|_{P_\alpha} &= \alpha^{-1}\left(\int_K \alpha(2|f(x)|)\frac{d\mu}{\mu(K)}\right)\\ &\geq \alpha^{-1}\left(\frac{1}{\mu(K)}\sum_{j=1}^{\infty} 2^j \alpha(t_j)\mu(K_j)\right) = \infty,\end{aligned}$$

which violates the Minkowski type inequality (2.11). In all, we have shown that there is a positive real number $c_2 > 1$ with

$$p_\alpha(t) \leq c_2$$

for almost every $t > 0$.

Similarly, we can find a positive constant $c_0 > 1$ for which

$$p_\beta(t) \leq c_0 \qquad \text{for almost every } t > 0, \tag{3.3}$$

that is to say,

$$\beta'(t) \leq c_0 \frac{\beta(t)}{t}. \tag{3.4}$$

From the identity (2.1), we get

$$x = \beta\left(\frac{x}{\alpha^{-1}(x)}\right) \quad \text{or} \quad \alpha(x) = \beta\left(\frac{\alpha(x)}{x}\right). \tag{3.5}$$

Hence we can notice that the condition (3.4) is equivalent to saying

$$\beta'\left(\frac{\alpha(t)}{t}\right) \leq c_0 t. \tag{3.6}$$

On the other hand, differentiate both sides of the identity (2.1) to have

$$\frac{\beta^{-1}(x)}{\alpha'(\alpha^{-1}(x))} + \frac{\alpha^{-1}(x)}{\beta'(\beta^{-1}(x))} = 1,$$

which is equivalent to

$$\frac{\beta^{-1}(\alpha(t))}{\alpha'(t)} + \frac{t}{\beta'(\alpha(t))} = 1. \tag{3.7}$$

Then the second identity of (3.5) leads to find

$$\beta^{-1} \circ \alpha(x) = \frac{\alpha(x)}{x}.$$

Therefore the identity (3.7) is equivalent to

$$\frac{y}{\alpha'(x)} + \frac{x}{\beta'(y)} = 1 \quad \text{with } \; y := \frac{\alpha(x)}{x}, \tag{3.8}$$

or

$$\alpha'(t) = s + t\frac{\alpha'(t)}{\beta'(s)}, \quad s = \frac{\alpha(t)}{t}. \tag{3.9}$$

Reflecting (3.6) to the identity (3.9), we have

$$\alpha'(t) \geq \frac{c_0}{c_0 - 1}\frac{\alpha(t)}{t},$$

which implies that

$$p_\alpha(t) \geq c_1$$

with $c_1 = \frac{c_0}{c_0-1}$. The proof is now completed. □

REMARK 3.2. *The Marcinkiewicz type interpolation theorem in* [6] *together with Theorem 3.1 implies that the Calderón-Zygmund operators are continuous on the corresponding new function spaces equipped with the conjugate pair of the convex Hölder base functions.*

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Department of Mathematics
Dankook University
119, Dandae-ro, Dongnam-gu
Cheonan-si, Chungnam, 31116
Republic of Korea
E-mail: hpak@dankook.ac.kr